Lecture 8

Thm If S is a hyperbolic R.S. and $f: S \to S$ is holo, then $J(f) = \emptyset$.
Moreover, either:

(a) every orbit converges to an attracting fixed point

(b) every orbit diverges to ∞; if S is open in $\hat{\mathbb{C}}$ and f extends to a neighbd of S in $\hat{\mathbb{C}}$, then $\exists\, \hat{z}$ in ∂S s.t.

$$f^n(z) \longrightarrow \hat{z} \quad \forall z \in S$$

(c) f auto of finite order

(d) f is conjugate to $z \mapsto e^{2\pi i \alpha} z$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and $S \simeq \mathbb{D}, \mathbb{D}^*, A_r$

Proof If we're not in (b), then $\exists\, \hat{z} = \lim_{k \to \infty} f^{n_k}(z)$ for some subsequence $n_k \to \infty$.

Consider $\left(f^{n_{k+1} - n_k}\right)_k$ which converges by Montel to some h s.t. $h(\hat{z}) = \hat{z}$.

<u>Case 1</u> if h contracts the Poincaré metric, then h has only one fixed point

Since f commutes with h,

$f(\hat{z}) = \hat{z}$

hence f is also strictly contracting near $\hat{z}$ and $\hat{z}$ is attracting fixed point.

Let A = attracting basin of $\hat{z}$

If $A \subsetneq S$, let z' be a pt in $S \setminus A$ closest to $\hat{z}$.

Then $d(f(\hat{z}), f(z')) < d(\hat{z}, z')$

So $f(z') \in A \Rightarrow z' \in A \Rightarrow$ <u>contr.</u>

<u>Case 2</u> h preserve Poincaré metric (hence so does f)

<u>Lemma 1</u> $\exists\, n_k \to \infty$ s.t. $f^{n_k} \to id$ on S, unif. on compact subsets.

<u>Cor.:</u> f is an auto of S

<u>Pf.:</u> If $f(x) = f(y) \Rightarrow f^n(x) = f^n(y)\ \forall n \geq 1$

$\Rightarrow x = y \Rightarrow f$ injective

<u>Pf of Lemma 1</u>

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{F,\,H} & \tilde{S} \\ \downarrow & & \downarrow \\ S & \xrightarrow{f,\,h} & S \end{array} \qquad f^{n_k} \to h$$

Lift h to $H : \tilde{S} \to \tilde{S}$ so that

$H(\tilde{z}) = \tilde{z}$ ($\tilde{z}$ lift of $\hat{z}$ s.t. $h(\hat{z}) = \hat{z}$)

$H : \mathbb{D} \to \mathbb{D}$ preserve P. metric, hence it is a rotation

Hence $\exists\, m_k \to \infty$ s.t. $H^{m_k} \to id_{\mathbb{D}}$

$\Rightarrow h^{m_k} \to id_S$

$\Rightarrow f^{n_k m_k} \to id_S$.

Lemma 2

If $f \in Aut(S)$ with $f^{n_k} \to id_S$, then either f has finite order or $S \simeq \mathbb{D}^*, \mathbb{D}, A_r = \{1 < |z| < r\}$.

Rmks if $S = \tilde{S}/\Gamma$, $S' = \tilde{S}'/\Gamma'$

1. $f : S \to S'$ lifts to $F : \tilde{S} \to \tilde{S}'$ so that for every $\gamma \in \Gamma$ there is $\gamma' \in \Gamma'$ s.t. $F(\gamma(z)) = \gamma'(F(z))$

   So, if $f^{n_k} \to id|_S$ then $\exists\, \gamma_k \subset \Gamma'$ s.t. $\gamma_k \circ F^{n_k} \to id|_{\tilde{S}}$

2. Let $\Gamma \subset G$ discrete subgroup of a top. group G. Then for each $\gamma \in \Gamma$ there exists a nbd N of id in G s.t. every $g \in N$ satisfies

$g\gamma g^{-1} \in \Gamma$ iff g commutes with γ.

Pf: $N\gamma N^{-1}$ is open nbd of γ in Γ if N is small, $N\gamma N^{-1} = \{\gamma\}$.

Pf of Lemma 2

$F_k = \gamma_k \circ F^{n_k} \longrightarrow id|_{\tilde{S}}$

for γ_k

For each $\gamma_0 \in \Gamma$ $\exists$ γ_j' s.t.

$$F_j \circ \gamma_0 = \gamma_j' \circ F_j$$

$F_j \to id|_{\tilde{S}}$ so by Rmk (2)

F_j commutes with γ_0

for large j. Two elts of Aut($\mathbb{D}$) commute iff they have same fixed points.

$\mathrm{Fix}(F_j) = \mathrm{Fix}(\gamma_0)$
for all j large
$\Rightarrow$ $\mathrm{Fix}(\gamma_0)$ DOES NOT depend on $\gamma_0 \in \Gamma$.
$\Rightarrow$ Γ is commutative subgroup of $\mathrm{Aut}(\mathbb{D})$
$\Rightarrow$ $\Gamma = \{1\}$ $\Rightarrow$ $S = \mathbb{D}$

$\Gamma \cong \mathbb{Z} = \langle \gamma \rangle$

γ parabolic $\Rightarrow S \simeq \mathbb{D}^*$

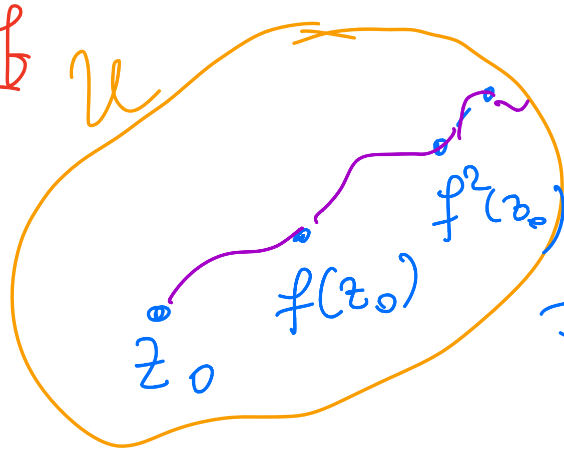
γ hyperbolic $\Rightarrow S \simeq A_r$.

Lemma 3

$\mathcal{U} \subset \hat{\mathbb{C}}$ hyperbolic and $f : \mathcal{U} \to \mathcal{U}$ extends to open nbd of $\overline{\mathcal{U}}$
Then if one orbit diverges, then all orbits converge

to the <u>same</u> $\hat{z} \in \partial \mathcal{U}$.

Pf



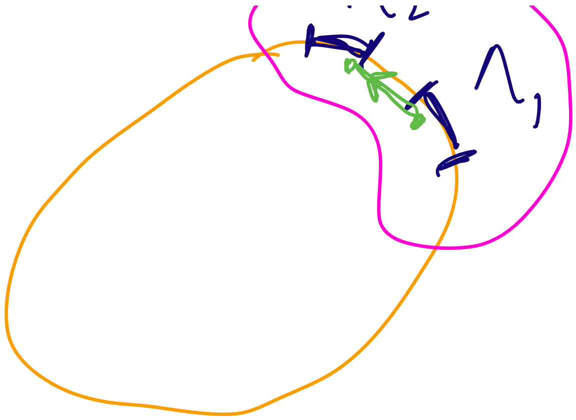
Let $p : [0,\infty) \to \mathcal{U}$ path with $f(p(t)) = p(t+1)$ $\forall t \in [0,\infty)$.

$$\delta_n := \operatorname{diam}_{\rho_{hyp}} p([n, n+1])$$

$$\delta_{n+1} \le \delta_n \le \delta_0. \quad \left(\begin{array}{c}\text{Schwarz}\\ \text{Pick}\end{array}\right)$$

Hence $\operatorname{diam}_{\rho_{sph}} p([n, n+1]) \to 0$

<u>Hence</u>; the set of acc. pts. of $\{p(t),\ t > 0\}$ is connected



Every elt of limit set is a fixed pt of f

$$f(p(t)) = p(t+1)$$

But since f extends across ∂S, the set of fixed pts of f on ∂S is discrete.

Hence, limit set is a pt $\{\hat{z}\}$ so $f^n(z) \longrightarrow \hat{z}$.

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